8.333 Fall 2025 Recitation 4: Kinetic theory recap

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These notes are largely a conglomeration of the previous years' recitation notes by Julien Tailleur, Amer Al-Hiyasat, and Sara Dal Cengio.

References. All the essential information in these recitations can be found in Chapter 3 of Mehran Kardar's *Statistical Physics of Particles*. Also see lectures 7-9 of his 8.333 OCW notes, and lectures 4-10 of Julien Tailleur's notes.

I. INTRO, DEFINITIONS AND NOTATIONS

We are concerned with extremely the high-dimensional problem of many-particle ($N \gtrsim 10^{23}$) Hamiltonian dynamics. How do we reduce the complicated microscopic dynamics to the simpler evolution of macroscopic quantities?

Work with N particles in 3 dimensions. Suppose particle i has position $\mathbf{q}_i = (q_1^x, q_1^y, q_1^z)$ and momentum $\mathbf{p}_i = (p_1^x, p_1^y, p_1^z)$. (On the blackboard, I use replace the boldfaced letter with the arrow version, i.e. $\mathbf{q}_i \to \vec{q}_i$.) Use the notation

$$\mathbf{Q} \equiv (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_N)\,, \qquad \mathbf{P} \equiv (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N)\,, \qquad \mathbf{\Gamma} \equiv (\mathbf{q}_1, \ldots, \mathbf{q}_N, \mathbf{p}_1, \ldots, \mathbf{p}_N)\,. \tag{1}$$

Also make the definitions

$$\mathbf{q}_{ij} \equiv \mathbf{q}_i - \mathbf{q}_j\,, \qquad q_{ij} \equiv |\mathbf{q}_{ij}|\,, \qquad p_i \equiv |\mathbf{p}_i|\,, \qquad d\Gamma_i \equiv d^3q_i d^3p_i\,. \tag{2}$$

The particles evolve under Hamiltonian dynamics with the Hamiltonian

$$H(\mathbf{Q}, \mathbf{P}) = \sum_{i=1}^{N} \left[\frac{p_i^2}{2m} + U(\mathbf{q}_i) + \frac{1}{2} \sum_{j \neq i} V(q_{ij}) \right] \equiv H_1(\mathbf{Q}, \mathbf{P}) + \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} V(q_{ij})\,. \tag{3}$$

In particular, we consider a two-body interaction potential $V(q)$ which is spherically symmetric, i.e. only depending on q rather than $\mathbf{q}$. Also define

$$U_i \equiv U(\mathbf{q}_i)\,, \qquad V_{ij} \equiv V(q_{ij})\,. \tag{4}$$

We will use the Poisson bracket, which for operators $A(\mathbf{Q}, \mathbf{P})$ and $B(\mathbf{Q}, \mathbf{P})$ is defined as

$$\{A, B\} \equiv \sum_{i=1}^{N} \left[\frac{\partial A}{\partial \mathbf{q}_i} \cdot \frac{\partial B}{\partial \mathbf{p}_i} - \frac{\partial A}{\partial \mathbf{p}_i} \cdot \frac{\partial B}{\partial \mathbf{q}_i} \right] = \frac{\partial A}{\partial \mathbf{Q}} \cdot \frac{\partial B}{\partial \mathbf{P}} - \frac{\partial A}{\partial \mathbf{P}} \cdot \frac{\partial B}{\partial \mathbf{Q}}\,. \tag{5}$$

It has the following properties (for operators A, B, C and scalar λ), which we will use:

$$\{B, A\} = -\{A, B\} \qquad \text{(antisymmetry)} \tag{6}$$

$$\{A, B + \lambda C\} = \{A, B\} + \lambda\{A, C\} \qquad \text{(bilinearity)} \tag{7}$$

$$\{A + \lambda C, B\} = \{A, B\} + \lambda\{C, B\} \qquad \text{(bilinearity)}\,. \tag{8}$$

The probability density over phase space $\rho(\mathbf{Q}, \mathbf{P}; t)$ is the probability density of particles at phase space point $(\mathbf{Q}, \mathbf{P})$ at time t. It evolves according to the Liouville equation, whose derivation proceeds as follows:

$$0 = \frac{d\rho}{dt} \qquad \text{(Liouville theorem)} \tag{9}$$

$$= \frac{\partial \rho}{\partial t} + \dot{\mathbf{Q}} \cdot \frac{\partial \rho}{\partial \mathbf{Q}} + \dot{\mathbf{P}} \cdot \frac{\partial \rho}{\partial \mathbf{P}} = \frac{\partial \rho}{\partial t} + \frac{\partial H}{\partial \mathbf{P}} \cdot \frac{\partial \rho}{\partial \mathbf{Q}} - \frac{\partial H}{\partial \mathbf{Q}} \cdot \frac{\partial \rho}{\partial \mathbf{P}} = \frac{\partial \rho}{\partial t} + \{H, \rho\} \tag{10}$$

$$= \frac{\partial \rho}{\partial t} + \{H_1, \rho\} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} \{V_{ij}, \rho\} \qquad \text{(Bilinearity of Poisson bracket)} . \tag{11}$$

The last term can be re-written by re-indexing and using the symmetry of $V_{ij} = V_{ji}$:

$$\frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} \{V_{ij}, \rho\} = \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} \sum_{k=1}^{N} \frac{\partial V_{ij}}{\partial \mathbf{q}_k} \cdot \frac{\partial \rho}{\partial \mathbf{p}_k} = \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} \left[\frac{\partial V_{ij}}{\partial \mathbf{q}_i} \cdot \frac{\partial \rho}{\partial \mathbf{p}_i} + \frac{\partial V_{ij}}{\partial \mathbf{q}_j} \cdot \frac{\partial \rho}{\partial \mathbf{p}_j} \right] = \sum_{i=1}^{N} \frac{\partial \rho}{\partial \mathbf{p}_i} \cdot \sum_{j \neq i} \frac{\partial V_{ij}}{\partial \mathbf{q}_i} . \tag{12}$$

Thus, we find the Liouville equation

$$\frac{\partial \rho}{\partial t} + \{\rho, H_1\} = \sum_{i=1}^{N} \frac{\partial \rho}{\partial \mathbf{p}_i} \cdot \sum_{j \neq i} \frac{\partial V_{ij}}{\partial \mathbf{q}_i} \qquad \textbf{(Liouville's equation)} \tag{13}$$

The left-hand side includes the one-body effects, such as advection due to the $\mathbf{P}$ and the flows under U. The right-hand side accounts for transfer of probability due to interactions. The Liouville equation is exact.

II. THE BBGKY HIERARCHY

The Liouville equation (13) for the probability density over the $6N$-dimensional phase space contains way too much information. We are interested in macroscopic quantities, like the average kinetic energy of the gas

$$\text{K.E.} \equiv \left\langle \frac{1}{N} \sum_{i=1}^{N} \frac{p_i^2}{2m} \right\rangle = \frac{1}{m} \langle p_1^2 \rangle = \frac{1}{m} \int \prod_{i=1}^{N} d\Gamma_i \rho(\mathbf{Q}, \mathbf{P}; t) p_1^2 \equiv \frac{1}{m} \int d\Gamma_1 \rho_1(\mathbf{q}_1, \mathbf{p}_1; t) p_1^2 , \tag{14}$$

where we have used the indistinguishability of the particles, and defined the 1-body probability density as the marginal probability density

$$\rho_1(\mathbf{q}_1, \mathbf{p}_1; t) \equiv \int \prod_{i=2}^{N} d\Gamma_i \rho(\mathbf{Q}, \mathbf{P}; t) . \tag{15}$$

Observables like Eq. (14) are one-body properties, which only require ρ_1, which is over a space of much lower dimension. Thus, it is sensible to look for the evolution of ρ_1.

Using the Liouville equation (13), we find

$$\frac{\partial \rho_1}{\partial t} = \int \prod_{i \geq 2} d\Gamma_i \rho(\mathbf{Q}, \mathbf{P}; t) = \int \prod_{i \geq 2} d\Gamma_i \Bigg[\underbrace{\{H_1, \rho\}}_{\equiv (1)} + \underbrace{\sum_{j=1}^{N} \frac{\partial \rho}{\partial \mathbf{p}_j} \cdot \sum_{k \neq j} \frac{\partial V_{jk}}{\partial \mathbf{q}_j}}_{\equiv (2)} \Bigg] . \tag{16}$$

Calculating each term individually, we have

$$(1) = \int \prod_{i \geq 2} d\Gamma_i \Bigg[\underbrace{\left(\frac{\partial H_1}{\partial \mathbf{q}_1} \cdot \frac{\partial \rho}{\partial \mathbf{p}_1} - \frac{\partial H_1}{\partial \mathbf{p}_1} \cdot \frac{\partial \rho}{\partial \mathbf{q}_1} \right)}_{\equiv (1a)} + \underbrace{\sum_{j \geq 2} \left(\frac{\partial H_1}{\partial \mathbf{q}_j} \cdot \frac{\partial \rho}{\partial \mathbf{p}_j} - \frac{\partial H_1}{\partial \mathbf{p}_j} \cdot \frac{\partial \rho}{\partial \mathbf{q}_j} \right)}_{\equiv (1b)} \Bigg] \tag{17}$$

Because $\partial H_1/\partial \mathbf{q}_1$ and $\partial H_1/\partial p_1$ only depend on $\mathbf{q}_1$ and $\mathbf{p}_1$, the integral over $\mathbf{q}_2, \mathbf{q}_3$, etc. and $\mathbf{p}_2, \mathbf{p}_3$, etc. passes through it, and we have

$$\text{(1a)} = \frac{\partial H_1}{\partial \mathbf{q}_1} \cdot \frac{\partial \rho_1}{\partial \mathbf{p}_1} - \frac{\partial H_1}{\partial \mathbf{p}_1} \cdot \frac{\partial \rho_1}{\partial \mathbf{q}_1} = \{H_1, \rho_1\} \; . \tag{18}$$

For part (1b), we use the fact that $\partial H_1/\partial \mathbf{q}_j$ doesn't depend on $\mathbf{p}_j$, and $\partial H_1/\partial \mathbf{p}_j$ doesn't depend on $\mathbf{q}_j$ to write

$$\text{(1b)} = \int \prod_{i \geq 2} d\Gamma_i \sum_{j \geq 2} \left[\frac{\partial}{\partial \mathbf{p}_j} \cdot \left(\frac{\partial H_1}{\partial \mathbf{q}_j} \rho \right) - \frac{\partial}{\partial \mathbf{q}_j} \cdot \left(\frac{\partial H_1}{\partial \mathbf{p}_j} \rho \right) \right] = 0 \; , \tag{19}$$

since the integral over a total derivative is zero (assuming there are no boundary terms, which is true for either periodic boundary conditions or a normalizeable ρ in open boundary conditions!).

Term (2), the interaction term, is also simplified by splitting the indices between $j = 1$ and $j > 1$:

$$\text{(2)} = \int \prod_{i \geq 2} d\Gamma_i \Bigg[\underbrace{\frac{\partial \rho}{\partial \mathbf{p}_1} \cdot \sum_{k \neq 1} \frac{\partial V_{1k}}{\partial \mathbf{q}_1}}_{\equiv \text{(2a)}} + \underbrace{\sum_{j \geq 2} \frac{\partial \rho}{\partial \mathbf{p}_j} \cdot \sum_{k \neq j} \frac{\partial V_{jk}}{\partial \mathbf{q}_j}}_{\equiv \text{(2b)}} \Bigg] \tag{20}$$

Term (2a) can be simplified using the indistinguishability of particles $k \neq 1$:

$$\text{(2a)} = \int \prod_{i \geq 2} d\Gamma_i \frac{\partial \rho}{\partial \mathbf{p}_1} \cdot \frac{\partial V_{12}}{\partial \mathbf{q}_1} \equiv (N-1) \int d\Gamma_2 \frac{\partial \rho_2}{\partial \mathbf{p}_1} \cdot \frac{\partial V_{12}}{\partial \mathbf{q}_1} \; , \tag{21}$$

where we have defined the *2-body* probability density

$$\rho_2(\mathbf{q}_1, \mathbf{q}_2, \mathbf{p}_1, \mathbf{p}_2; t) \equiv \int \prod_{i \geq 3} d\Gamma_i \rho(\mathbf{Q}, \mathbf{P}; t) \; . \tag{22}$$

Finally, term (2b) is zero for the same reason as term (1b) (19):

$$\text{(2b)} = \int \prod_{i \geq 2} d\Gamma_i \sum_{j \geq 2} \frac{\partial}{\partial \mathbf{p}_j} \cdot \left(\rho \sum_{k \neq j} \frac{\partial V_{jk}}{\partial \mathbf{q}_j} \right) = 0 \; . \tag{23}$$

Thus, we find the overall 1-body evolution equation

$$\frac{\partial \rho_1}{\partial t} + \{\rho_1, H_1\} = (N-1) \int d\Gamma_2 \frac{\partial \rho_2}{\partial \mathbf{p}_1} \cdot \frac{\partial V_{12}}{\partial \mathbf{q}_1} \; . \tag{24}$$

This contains much less information than the Liouville equation (13). It is *almost* closed in ρ_1, but has the annoying ρ_2-dependence on the right-hand side. Intuitively, this is because the probability density of a single particle can't be understood without accounting for the joint probability density of it encountering another particle. Unfortunately, $\rho_2(\mathbf{q}_1, \mathbf{q}_2, \mathbf{p}_1, \mathbf{p}_2) \neq \rho_1(\mathbf{q}_1, \mathbf{p}_1)\rho_1(\mathbf{q}_2, \mathbf{p}_2)$ since the particles are not independent. For example, for repulsive interactions, $\rho_2(\mathbf{q}, \mathbf{q}, \mathbf{p}, \mathbf{p}') < \rho_1(\mathbf{q}, \mathbf{p})\rho_1(\mathbf{q}, \mathbf{p}')$ since having one particle at location $\mathbf{q}$ makes it less likely to have another particle there.

To find the evolution of ρ_2, we can make a similar calculation to Eqs. (16)-(24). Sparing you the details, the final answer is

$$\frac{\partial \rho_2}{\partial t} + \{\rho_2, H_1 + V_{12}\} = (N-2) \int d\Gamma_3 \left[\frac{\partial \rho_3}{\partial \mathbf{p}_1} \cdot \frac{\partial V_{13}}{\partial \mathbf{q}_1} + \frac{\partial \rho_3}{\partial \mathbf{p}_2} \cdot \frac{\partial V_{23}}{\partial \mathbf{q}_2} \right] . \tag{25}$$

The 2-body equation contains dependence on the 3-body density. Likewise, the evolution of the 3-body density will depend on the 4-body density, and so on. This is the BBGKY hierarchy. Because we are only interested in macroscopic, few-body obervables, we must truncate this hierarchy somewhere, using some physically-motivated approximation.

III. THE BOLTZMANN EQUATION

Let's define the number densities

$$f_1(\mathbf{q}_1, \mathbf{p}_1, t) \equiv N\rho_1(\mathbf{q}_1, \mathbf{p}_1; t) \tag{26}$$

$$f_2(\mathbf{q}_1, \mathbf{q}_2, \mathbf{p}_1, \mathbf{p}_2, t) \equiv N(N-1)\rho_2(\mathbf{q}_1, \mathbf{q}_2, \mathbf{p}_1, \mathbf{p}_2; t) \tag{27}$$

$$\vdots$$

$$f_s(\mathbf{q}_1, \ldots, \mathbf{q}_s, \mathbf{p}_1, \ldots, \mathbf{p}_s, t) \equiv \frac{N!}{(N-s)!}\rho_s(\mathbf{q}_1, \ldots, \mathbf{q}_s, \mathbf{p}_1, \ldots, \mathbf{p}_s; t) \ . \tag{28}$$

These are no longer probability densities. The normalization condition for f_1 is, for example,

$$\int d\Gamma_1 f_1(\mathbf{q}_1, \mathbf{p}_1, t) = N \ . \tag{29}$$

Now let's write out the 2-body equation for $f_2(\mathbf{q}_1, \mathbf{q}_2, \mathbf{p}_1, \mathbf{p}_2)$ explicitly:

$$\dot{f}_2 + \underbrace{\frac{\partial f_2}{\partial \mathbf{q}_1} \cdot \frac{\mathbf{p}_1}{m} + \frac{\partial f_2}{\partial \mathbf{q}_2} \cdot \frac{\mathbf{p}_2}{m}}_{\equiv (1)} - \underbrace{\left[\frac{\partial f_2}{\partial \mathbf{p}_1} \cdot \frac{\partial U_1}{\partial \mathbf{q}_1} + \frac{\partial f_2}{\partial \mathbf{p}_2} \cdot \frac{\partial U_2}{\partial \mathbf{q}_2}\right]}_{\equiv (2)} - \underbrace{\left[\frac{\partial f_2}{\partial \mathbf{p}_1} - \frac{\partial f_2}{\partial \mathbf{p}_2}\right] \cdot \frac{\partial V_{12}}{\partial \mathbf{q}_1}}_{\equiv (3)} = \underbrace{\int d\Gamma_3 \left[\frac{\partial f_3}{\partial \mathbf{p}_1} \cdot \frac{\partial V_{13}}{\partial \mathbf{q}_1} + \frac{\partial f_3}{\partial \mathbf{p}_2} \cdot \frac{\partial V_{23}}{\partial \mathbf{q}_2}\right]}_{\equiv (4)} , \tag{30}$$

where we have used the fact that $\partial V_{12}/\partial \mathbf{q}_2 = -\partial V_{12}/\partial \mathbf{q}_1$.

We will now use dimensional analysis to guess which terms from this equation are the most important. For a gas at room temperature, there are a convenient series of scale separation that make this easy. (This is where the applicability of these calculations to other many situations—e.g. astrophysics—breaks down, since long-range interactions and higher densities mess things up.)

Air molecules at room temperature have typical velocities of $v \approx 10^2 m/s$ and interaction radii of $d \approx 10^{-10} m$. Thus, the time it takes a collision to occur $\tau_c \approx d/v \approx 10^{-12} s$ is very small compared to, say, the time it takes a molecule to cross a box $U(\mathbf{q})$ of length $1m$, $\tau_U \approx L/v \approx 10^{-2} s$. The density of air is also very low: $n \equiv N/V \approx 10^{26}/m^3 \ll 1/d^3$. Thus, the distance a particle typically travels between collisions, ℓ_{MF} or the "mean-free path", is large compared to d. This can be estimated by considering the volume $\ell_{\text{MF}} \pi d^2$ swept out by a particle traveling this distance, and comparing it to the typical volume one must search before encountering a particle, V/N:

$$\ell_{\text{MF}} \pi d^2 \approx \frac{V}{N} \quad \Longrightarrow \quad \ell_{\text{MF}} \approx \frac{1}{nd^2} \ . \tag{31}$$

This is given by $\ell_{\text{MF}} \approx 10^{-6} m$. The mean-free time is then given by $\tau_{\text{MF}} = \ell_{\text{MF}}/v \approx 10^{-8} m$.

We have found three processes, each well-separated from the other in terms of length and time-scales:

$$\tau_c \ll \tau_{\text{MF}} \ll \tau_U \ , \qquad d \ll \ell_{\text{MF}} \ll \ell_U \ . \tag{32}$$

These are summarized by the following table:

Process	Length scale	Time scale
Collisions	$d \approx 10^{-10} m$	$\tau_c \approx 10^{-12} s$
Free motion between collisions	$\ell_{\text{MF}} \approx 10^{-6} m$	$\tau_{\text{MF}} \approx 10^{-8} s$
Effects of $U(\mathbf{q})$	$\ell_U \approx 1m$	$\tau_U \approx 10^{-2} s$

The Boltzmann equation, which we will now derive, exploits these two separations of length and time scale.

Now let's return to Eq. (30) and examine it term-by-term. All terms have dimension $T^{-1} N^2 L^{-6}$. Let V, U, and KE indicate the energy scales of $V(q)$, $U(\mathbf{q})$, and $p_i^2/2m$ respectively. Also suppose that the system size is comparable to

ℓ_U, so that $f_2 \sim (N/\ell_U^3)^2$. Finally, define a new "length scale of interest" $\ell \ll \ell_U$, such that $\partial f_2/\partial \mathbf{q}_i \sim f_2/\ell$. We find the approximate scaling of each term

$$\textcircled{1} \quad \sim \quad \text{KE}\left(\frac{N}{\ell_U^3}\right)^2 \frac{1}{\ell}\frac{1}{mv} \tag{33}$$

$$\textcircled{2} \quad \sim \quad U\frac{1}{\ell_U}\left(\frac{N}{\ell_U^3}\right)^2 \frac{1}{mv} \quad \sim \quad \frac{U}{\text{KE}}\frac{\ell}{\ell_U}\textcircled{1} \quad \ll \quad \textcircled{1} \tag{34}$$

$$\textcircled{3} \quad \sim \quad V\frac{1}{d}\left(\frac{N}{\ell_U^3}\right)^2 \frac{1}{mv} \tag{35}$$

$$\textcircled{4} \quad \sim \quad \int V\frac{1}{d}\left(\frac{N}{\ell_U^3}\right)^3 \sim V d^3 \frac{1}{d}\left(\frac{N}{\ell_U^3}\right)^3 \frac{1}{mv} \quad \sim \quad N\frac{d^3}{\ell_U^3}\textcircled{3} \quad \ll \quad \textcircled{3} \tag{36}$$

We can thus eliminate term ②, since the gradients of the external potential are chosen to be significantly smaller than those of f_2 (and the potential energy U is at most comparable with the kinetic energy). We can also, crucially, eliminate term ④, since it is smaller than term ③ by a factor of $nd^3 \sim 10^{-4} \ll 1$. Since ④ contains all the f_3-dependence, we have thus truncated the BBGKY hierarchy.

We are left with the new equation

$$\dot{f}_2 = \left[\frac{\partial f_2}{\partial \mathbf{p}_1} - \frac{\partial f_2}{\partial \mathbf{p}_2}\right] \cdot \frac{\partial V_{12}}{\partial \mathbf{q}_1} - \frac{\partial f_2}{\partial \mathbf{q}_1} \cdot \frac{\mathbf{p}_1}{m} - \frac{\partial f_2}{\partial \mathbf{q}_2} \cdot \frac{\mathbf{p}_2}{m} \,, \tag{37}$$

where the $=$ sign should really be an $\approx$ but we will (semi-phenomenologically) pretend the strict equality holds from now on.

Let's simplify $\partial f_2/\partial \mathbf{q}_i$ further. We can change the coordinates $\mathbf{q}_1$, $\mathbf{q}_2$ to $\mathbf{q}_+ \equiv (\mathbf{q}_1 + \mathbf{q}_2)/2$ and $\mathbf{q} \equiv \mathbf{q}_1 - \mathbf{q}_2$, and note that (suppressing the $\mathbf{p}$ dependence)

$$\frac{\partial f_2}{\partial \mathbf{q}_1} = 2\frac{\partial f_2}{\partial \mathbf{q}_+} + \frac{\partial f_2}{\partial \mathbf{q}} \,, \qquad \frac{\partial f_2}{\partial \mathbf{q}_2} = 2\frac{\partial f_2}{\partial \mathbf{q}_+} - \frac{\partial f_2}{\partial \mathbf{q}} \,. \tag{38}$$

Since the gradient f_2 with respect to $\mathbf{q}$ is of the order $1/d$ while variations with respect to $\mathbf{q}_+$ are the inverse of a meso- or macroscopic lengthscale (e.g. $\sim 1/\ell$), we can neglect the $\partial/\partial \mathbf{q}_+$ terms, and approximate

$$\frac{\partial f_2}{\partial \mathbf{q}_1} \approx \frac{\partial f_2}{\partial \mathbf{q}} \,, \qquad \frac{\partial f_2}{\partial \mathbf{q}_2} \approx -\frac{\partial f_2}{\partial \mathbf{q}} \qquad \Longrightarrow \qquad \frac{\partial f_2}{\partial \mathbf{q}_1} \cdot \frac{\mathbf{p}_1}{m} + \frac{\partial f_2}{\partial \mathbf{q}_2} \cdot \frac{\mathbf{p}_2}{m} \approx \frac{\partial f_2}{\partial \mathbf{q}} \cdot \left(\frac{\mathbf{p}_1}{m} - \frac{\mathbf{p}_2}{m}\right) \,. \tag{39}$$

Return to the 1-body equation (24), which in terms of f_1 and f_2 is given by

$$\frac{\partial f_1}{\partial t} + \{f_1, H_1\} = \int d\Gamma_2 \frac{\partial f_2}{\partial \mathbf{p}_1} \cdot \frac{\partial V_{12}}{\partial \mathbf{q}_1} \equiv \left.\frac{\partial f_1}{\partial t}\right|_{\text{coll.}} \,. \tag{40}$$

In the steady state, Eqs. (37) and (39) gives us

$$\left[\frac{\partial f_2}{\partial \mathbf{p}_1} - \frac{\partial f_2}{\partial \mathbf{p}_2}\right] \cdot \frac{\partial V_{12}}{\partial \mathbf{q}_1} = \frac{\partial f_2}{\partial \mathbf{q}} \cdot \left(\frac{\mathbf{p}_1}{m} - \frac{\mathbf{p}_2}{m}\right) \tag{41}$$

$$\Longrightarrow \int d\Gamma_2 \left[\frac{\partial f_2}{\partial \mathbf{p}_1} - \frac{\partial f_2}{\partial \mathbf{p}_2}\right] \cdot \frac{\partial V_{12}}{\partial \mathbf{q}_1} = \int d\Gamma_2 \frac{\partial f_2}{\partial \mathbf{p}_1} \cdot \frac{\partial V_{12}}{\partial \mathbf{q}_1} = \int d\Gamma_2 \frac{\partial f_2}{\partial \mathbf{q}} \cdot \left(\frac{\mathbf{p}_1}{m} - \frac{\mathbf{p}_2}{m}\right) \,. \tag{42}$$

The first equality in Eq. (42) is obtained by noting that the second term is a total derivative in $\mathbf{p}_2$, which is integrated over. Thus, the second equality of Eq. (42) allows us to replace the right-hand side of Eq. (40), yielding

$$\left.\frac{\partial f_1}{\partial t}\right|_{\text{coll.}} = \frac{1}{m}\int d^3\mathbf{q}\, d^3\mathbf{p}_2 \frac{\partial f_2}{\partial \mathbf{q}} \cdot (\mathbf{p}_1 - \mathbf{p}_2) \,. \tag{43}$$

Keep in mind that $\mathbf{q} = \mathbf{q}_1 - \mathbf{q}_2$ is the separation between coordinates 1 and 2.